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A Multi-curve Random Field LIBOR Market Model

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Background

- Recent development in interest rate modeling since HJM:
- Libor Market Model by Brace, Gatarek, Musiela (MF'97), Miltersen, Sandmann and Sondermann (JF '97)
- Random Field Model by Kennedy (MF '94, '97) Goldstein (RFS 2013)
- Random Field Libor Market Model (Wu and Xu 2013) merge the above two approach
- In this work, we extend the RFLMM of Wu and Xu (2013) to the multi-curve setting.

Motivation: Inconsistency between Similar Rates since the Crisis

- Before August 2007, there was consistency between similar rates: The spreads between similar rates were within a few basis points and were regarded as negligible
- After the August 2007 credit crisis: The advent of the crisis widened the basis. It became more severe as the crisis deepened.

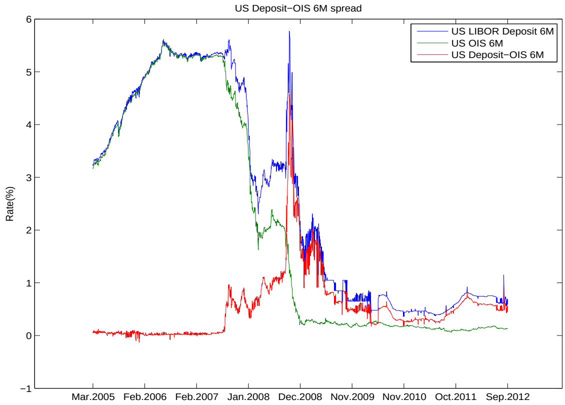

Figure: U.S. LIBOR Deposit-6M(spot) rates vs U.S. OIS-6M rates. Quotations Mar.15.2005-Sep.14.2012(source: Bloomberg)

Multi-curve Pricing Methodology

Similar rates are modeled jointly but distinctly, for example,

- the rates for generating future cash flows
- the rates for discounting

Dynamics of Multi-curve LIBOR Rate in the Random Field Setting

We consider the two-curve setting, the dynamics of rates for discounting under the T_k forward measure:
$L_k(t) := L(t, T_{k-1}, T_k) = \frac{1}{\delta}[\frac{P(t,T_{k-1})}{P(t,T_k)} - 1]$

$$dL_k(t) = L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t,u) dW^{T_k}(t,u) du$$

with

$$corr[dW^{T_k}(t,T_1), dW^{T_k}(t,T_2)] = c_d(t,T_1,T_2)$$

We also need to model the evolution of the FRA rate $F_k(t)$, for generating future cash flows:

$$dF_k(t) = F_k(t) \int_{T_{k-1}}^{T_k} \eta_k(t,u) dB^{T_k}(t,u) du$$

with

$$corr[dB^{T_k}(t,T_1), dB^{T_k}(t,T_2)] = c_f(t,T_1,T_2)$$

and

$$corr[dW^{T_k}(t,T_1), dB^{T_k}(t,T_2)] = c_{df}(t,T_1,T_2)$$

Theorem (Random field dynamics under forward measures (Two-curve))

The dynamics of the instantaneous forward rates $L_k(t)$ and FRA rates $F_k^f(t)$ under the T_j-forward measure for $j < k$ is

$$dL_k(t) = L_k(t) \int_{T_{k-1}}^{T_k} \xi_k(t,u)[dW^{T_j}(t,u) + \Lambda_j^k(t,u)dt]du,$$

$$dF_k^f(t) = F_k^f(t) \int_{T_{k-1}}^{T_k} \eta_k(t,u)[dB^{T_j}(t,u) + \Lambda_j'^k(t,u)dt]du,$$

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Theorem (Two-curve random field dynamics under forward measures(Cont'd))

with

$$\Lambda_j^k(t,u) = \sum_{i=j+1}^{k} \int_{T_{i-1}}^{T_i} \frac{\delta_i L_i(t)\xi_i(t,v)c_d(t,v,u)}{\delta_i L_i(t)+1} dv,$$

and

$$\Lambda_j^{\prime k}(t,u) = \sum_{i=j+1}^{k} \int_{T_{i-1}}^{T_i} \frac{\delta_i L_i(t)\xi_i(t,v)c_{df}(t,v,u)}{\delta_i L_i(t)+1} dv,$$

where $W^{T_j}(t,u)$ is a random field under T_j-forward measure. The above equations admit a unique strong solution if the coefficient $\xi_k(\cdot,\cdot)$ are locally bounded, locally Lipschitz continuous and predictable.

Estimation

Use historical time series to estimate the parameters of the model.

- We use unscented Kalman filter for parameter estimation.
- We then investigate the pricing and hedging performance for different models:
  1)single-curve LMM;
  2)two-curve LMM;
  3)single-curve RFLMM;
  4)two-curve RFLMM.

The Construction of Different Curves

We can build the different curves as follows.

- LIBOR standard curve: bootstrapped from short term LIBOR deposits(below 1 year), mid-term FRA on LIBOR 3M (between 1-2 year) and mid/long-term swaps on LIBOR 6M (beyond 2 years).
- OIS curve: bootstrapped from the U.S. OIS rates.
- LIBOR 6M curve: the LIBOR 6M curve bootstrapped from the LIBOR deposit 6M, mid-term FRA on LIBOR-6M(up to 2 years)and mid/long-term swaps on LIBOR 6M (beyond 2 year).

- single-curve modeling:
  discount curve: LIBOR standard curve.
  the curve for generating future cash flows: LIBOR standard curve.
- two-curve modeling:
  discount curve : the OIS curve.
  the curve for generating future cash flows: LIBOR 6M.

We need to specify the instantaneous volatility $\xi(t, T_k)$ and the correlation structure $c(t, x, y)$.

The Instantaneous Volatility

Following Rebonato (1999), the instantaneous volatility takes the form:

$$\xi(t, T_k) = [a + b(T_k - t)]e^{-c(T_k - t)} + d; a, b, c, d > 0$$

.

- the function is flexible enough to be able to produce either a hump-shaped or monotonically decreasing instantaneous volatility.
- analytical integration of the function's square allows fast calculation of forward rate variance and covariance.

The Correlation Structure

For traditional LMM, the instantaneous correlation between forward rates $L_i(t)$ and $L_j(t)$ is defined as

$$\frac{\frac{d}{dt}\langle L_i, L_j\rangle(t)}{\sqrt{\frac{d}{dt}\langle L_i, L_i\rangle(t)\frac{d}{dt}\langle L_j, L_j\rangle(t)}} = \rho_{i,j}(t),$$

We follow Coffey and Schoenmakers (2000) to specify $\rho_{i,j} = e^{-\frac{|i-j|}{N-1}\left(\rho_\infty + \rho_0 \frac{N-i-j+1}{N-2}\right)}$.

For the random field LMM, the instantaneous correlation between forward rates $L_i(t)$ and $L_j(t)$ is given by

$$\frac{\frac{d}{dt}\langle L_i, L_j\rangle(t)}{\sqrt{\frac{d}{dt}\langle L_i, L_i\rangle(t)\frac{d}{dt}\langle L_j, L_j\rangle(t)}} = \frac{\int_{T_{i-1}}^{T_i}\int_{T_{j-1}}^{T_j}\xi_i(t,x)\xi_j(t,y)c(t,x,y)dxdy}{\sqrt{\iint_{T_{i-1}}^{T_i}\xi_i(t,x)\xi_i(t,y)c(t,x,y)dxdy\iint_{T_{j-1}}^{T_j}\xi_j(t,x)\xi_j(t,y)c(t,x,y)dxdy}}$$

we only need to take

$$c(t,x,y) = e^{-\rho_\infty|x-y|} := c(x,y).$$

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Table: Pricing formulas for caps and swaptions

caps, single-curve	$\tau P(t, T_k)\mathbf{Bl}(K, L_k(t), \sigma_k\sqrt{T_{k-1}-t})$
caps, multi-curve	$\tau P(t, T_k)\mathbf{Bl}(K, F_k(t), \overline{\sigma}_k\sqrt{T_{k-1}-t})$
swapts, single-curve	$\sum_{k=i+1}^{j}\tau P(t, T_k)\mathbf{Bl}(K, S_{i,j}(t), \sigma_{i,j}\sqrt{T_{k-1}-t})$
swapts, multi-curve	$\sum_{k=i+1}^{j}\tau P(t, T_k)\mathbf{Bl}(K, \overline{S}_{i,j}(t), \overline{\sigma}_{i,j}\sqrt{T_{k-1}-t})$

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Table: Black implied volatility for caps and swaptions, single-curve

caps, BM $\sqrt{\frac{1}{T_{k-1}-t}\int_t^{T_{k-1}} \xi_k^2(s)ds}$

caps, RF $\sqrt{\frac{1}{T_{k-1}-t}\int_t^{T_{k-1}}[\int_{T_{k-1}}^{T_k}\int_{T_{k-1}}^{T_k} \xi_k(t,x)\xi_k(t,y)c(x,y)dxdy]dt}.$

swpts, BM $\sqrt{\frac{1}{T_i-t}\sum_{l,k=i+1}^{j}\Phi_k(t)\Phi_l(t)\int_t^{T_i}\rho_{k,l}\xi_k(s)\xi_l(s)ds},$

swpts, RF $\sqrt{\frac{1}{T_i-t}\sum_{l,k=i+1}^{j}\Phi_k(t)\Phi_l(t)\int_t^{T_i}\int_{T_{k-1}}^{T_k}\int_{T_{l-1}}^{T_l}\xi_k(t,x)\xi_l(t,y)c(x,y)dxdyds},$

with $\Phi_k(t) = \frac{\delta_k L_k(t)\gamma_k^{i,j}(t)}{1+\delta_k L_k(t)}$

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Table: Black implied volatility for caps and swaptions, multi-curve

caps, BM	$\sqrt{\frac{1}{T_{k-1}-t}\int_t^{T_{k-1}} \eta_k^2(s)ds}$
caps, RF	$\sqrt{\frac{1}{T_{k-1}-t}\int_t^{T_{k-1}}[\int_{T_{k-1}}^{T_k}\int_{T_{k-1}}^{T_k} \eta_k(t,x)\eta_k(t,y)c(x,y)dxdy]dt}.$
Swpts,BM	$\sqrt{\frac{1}{T_i-t}\int_t^{T_i}\lVert\sum_{k=i+1}^{j}[\frac{\alpha_k^{i,j}(s)\delta_k L_k(s)\xi_k(s)}{1+\delta_k L_k(s)}dW^{T_k}(s) + \frac{\beta_k^{i,j}(s)\delta_k L_k(s)\eta_k(s)}{1+\delta_k L_k(s)}dB^{T_k}(s)]\rVert^2 ds}$
Swpts, RF	$\sqrt{\frac{1}{T_i-t}\int_t^{T_i}\lVert\sum_{k=i+1}^{j}\int_{T_k-1}^{T_k}[\frac{\alpha_k^{i,j}(s)\delta_k L_k(s)\xi_k(s,u)}{1+\delta_k L_k(s)}dW^{T_k}(s,u) + \frac{\beta_k^{i,j}(s)\delta_k L_k(s)\eta_k(s,u)}{1+\delta_k L_k(s)}dB^{T_k}(s,u)]du\rVert^2 ds}$

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Table: Unscented Kalman Filter Estimation Results, July. 9, 2007–Oct. 14, 2008

Model	curve	a	b	c	d	ρ_∞	ρ_0
single-curve LMM		0.5543	1.2378	5.6365	1.275	2.7952	3.5513
		(0.022)	(0.0265)	(0.0763)	(0.0123)	(0.0124)	(0.1568)
single-curve RFLMM		0.5394	1.3457	5.4772	1.4685	8.9834	-
		(0.003)	(0.0446)	(0.167)	(0.0476)	(0.0252)	-
two-curve LMM	discounting	0.5546	1.7655	5.8773	0.9754	2.6553	3.3572
		(0.032)	(0.0334)	(0.203)	(0.0104)	(0.0232)	(0.1292)
	forwarding	0.5523	1.2483	5.4772	1.4582	2.7834	3.527
		(0.032)	(0.0333)	(0.0693)	(0.0584)	(0.082)	(0.0632)
two-curve RFLMM	discounting	0.5564	1.7834	5.9823	0.8468	8.3834	-
		(0.022)	(0.0634)	(0.332)	(0.0592)	(0.0134)	-
	forwarding	0.5342	1.3452	5.3343	1.4646	8.9652	-
		(0.016)	(0.0393)	(0.224)	(0.0633)	(0.0124)	-

Pricing Performance

The pricing procedure:

- Estimate the parameters from the time series of underlying interest rates.
- Compute model prices of caps and swaptions using the estimated parameters.
- Compare the difference of model price and market price.

$$RMSE = \sqrt{\frac{\sum e_i^2}{M}}$$

We perform in-sample and out-of-sample pricing for the four different models,1) single-curve LMM, 2)multi-curve LMM, 3)single-curve RFLMM and 4)multi-curve RFLMM.

The Pricing Performance of Different Models

- In-sample pricing: Use the estimation results to price the instrument in the same period(Jul.9,07-Oct.14,08).
- Out-of-sample pricing: Use the estimatio results to price the instrument in a later period(Oct.15,08-Oct.9,09).

Table: Average RMSE Pricing Errors of European Swaptions(%)

	In-sample Errors	Out-of-sample Errors
single-curve LMM	2.38	2.67
two-curve LMM	2.15	2.32
single-curve RFLMM	1.27	1.38
two-curve RFLMM	0.85	0.95

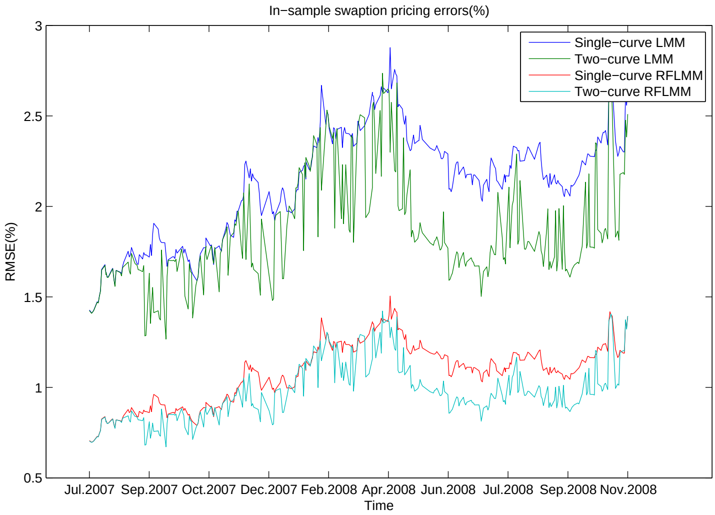

Figure: Time series of RMSE of Swaptions for single-curve LMM, RFLMM and two-curve LMM, RFLMM over the period Jul.07-Oct.08(In-sample pricing)

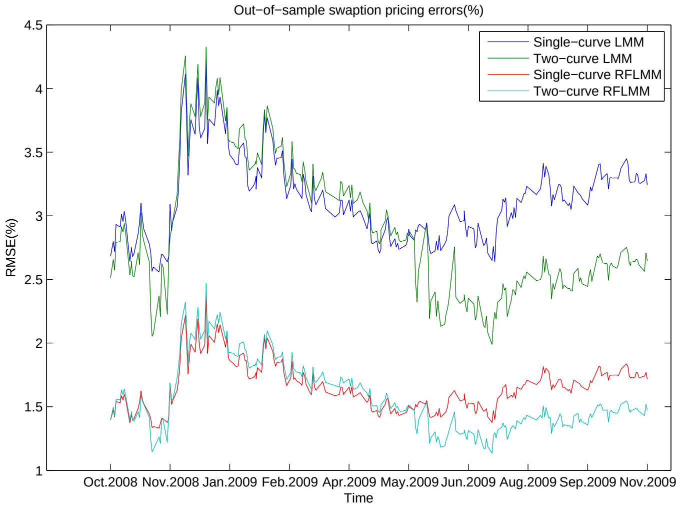

Figure: Time series of RMSE of Swaptions for single-curve LMM, RFLMM and two-curve LMM, RFLMM over the period Oct.08-Oct.09(Out-of-sample pricing)

RMSE of Caps and Swaptions by Maturity

- We can examine the pricing errors for individual maturity caps or swaptions, beyond the overall RMSES.

Table: ATM Caplet Valuation RMSEs(Jul.9,07-Oct.14,08)

	1	2	3	4	5	6	7	8	9
single-curve	0.5047	0.4846	0.4746	0.4665	0.4575	0.4479	0.4176	0.3945	0.3532
LMM	(0.2516)	(0.1749)	(0.1427)	(0.1214)	(0.1014)	(0.0927)	(0.0893)	(0.0893)	(0.0885)
two-curve	0.4866	0.4656	0.4548	0.4454	0.4225	0.4012	0.3760	0.3553	0.3312
LMM	(0.1596)	(0.1658)	(0.1383)	(0.1192)	(0.1085)	(0.0942)	(0.0910)	(0.0914)	(0.0899)
single-curve	0.3458	0.3137	0.2909	0.2710	0.2521	0.2346	0.2005	0.1708	0.1337
RFLMM	(0.2512)	(0.1412)	(0.1280)	(0.1167)	(0.1028)	(0.0979)	(0.0962)	(0.0992)	(0.0998)
two-curve	0.3256	0.2931	0.2698	0.2490	0.2166	0.1858	0.1557	0.1261	0.1184
RFLMM	(0.1153)	(0.1264)	(0.1200)	(0.1118)	(0.1065)	(0.0975)	(0.0965)	(0.0999)	(0.1001)

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Table: ATM Swaption Valuation RMSEs for two-curve RFLMM(Jul.9,07-Oct.14,08)

maturities (years)	lengths(year) 1	2	3	4	5	6	7	8	9
0.5	1.0785 (0.1129)	1.0707 (0.0572)	1.0695 (0.0358)	1.0711 (0.0255)	1.0751 (0.0194)	1.0762 (0.0161)	1.0790 (0.0139)	1.0809 (0.0122)	1.0829 (0.0109)
1	1.0228 (0.1123)	1.0220 (0.0615)	1.0214 (0.0415)	1.0247 (0.0311)	1.0289 (0.0252)	1.0317 (0.0210)	1.0348 (0.0186)	1.0375 (0.0165)	1.0405 (0.0149)
2	0.9368 (0.0993)	0.9409 (0.0628)	0.9455 (0.0463)	0.9518 (0.0380)	0.9581 (0.0315)	0.9613 (0.0279)	0.9651 (0.0247)	0.9692 (0.0223)	
3	0.8727 (0.0906)	0.8817 (0.06110	0.8886 (0.0509)	0.8932 (0.0415)	0.8999 (0.0359)	0.9050 (0.0314)	0.9105 (0.0280)		
4	0.8202 (0.0880)	0.8287 (0.07660	0.8361 (0.05750	0.8443 (0.0473)	0.8480 (0.0402)	0.8547 (0.0353)			
5	0.7716 (0.2296)	0.7820 (0.0958)	0.7921 (0.0656)	0.7961 (0.0524)	0.8041 (0.0444)				
7	0.6903 (0.1063)	0.6972 (0.0752)	0.7052 (0.0603)						

Hedging Performance

The idea of hedging: creating a portfolio whose value goes in opposite direction than the value of instruments when market fluctuates.

Delta hedging:

$$\Delta = \frac{\partial \mathbf{Cplt}(L_k(t))}{\partial L_k(t)} = \frac{\mathbf{Cplt}(L_k(t)+h) - \mathbf{Cplt}(L_k(t)-h)}{2h},$$

$$\Delta = \frac{\partial \mathbf{Swpt}(S_{i,j}(t))}{\partial S_{i,j}(t)} = \frac{\mathbf{Swpt}(S_{i,j}(t)+h) - \mathbf{Swpt}(S_{i,j}(t)-h)}{2h}$$

We use Hedging Variance Ratio (HVR) to examine the hedging performance of models.

HVR is computed as follows:

- At time t_1, Hedging error:
  $\text{HedgingError}_{t_1} = V(t_2) - V(t_1) - \Delta_{t_1}[F(t_2) - F(t_1)]$.
- Accumulated Hedging Error:
  $\text{AccuHedgingError} = \sum_{k=1}^{m}[V(t_{k+1}) - V(t_k) - \Delta_{t_k}[F(t_{k+1}) - F(t_k)]]$.
- $HVR = 1 - \frac{Var(AccuHedgingError)}{Var(V)}$

Table: Hedging performance of caplets and swaptions

	Average HVR for Four Models(%)			
	single-curve LMM	two-curve LMM	single-curve RFLMM	two-curve RFLMM
caps	0.9983	0.9985	0.9988	0.9989
swaptions	0.9724	0.9782	0.9820	0.9860

Table: Hedging performance (HVR) of ATM Caplets

	1	2	3	4	5	6	7	8	9	10
single-curve LMM	0.9999	0.9977	0.9995	0.9993	0.9989	0.9986	0.9984	0.9979	0.9937	0.9954
two-curve LMM	0.9998	0.9991	0.9996	0.9995	0.9990	0.9987	0.9984	0.9979	0.9952	0.9951
single-curve RFLMM	0.9999	0.9976	0.9996	0.9996	0.9992	0.9990	0.9987	0.9983	0.9978	0.9952
two-curve RFLMM	0.9998	0.9991	0.9996	0.9996	0.9992	0.9989	0.9986	0.9982	0.9977	0.9950

Table: Hedging performance (HVR) of ATM Swaptions for two-curve RFLMM

	1	2	3	4	5	6	7	8	9
0.5	0.9969	0.9867	0.9867	0.9893	0.9853	0.9748	0.9964	0.9912	0.9815
1	0.9831	0.9854	0.9731	0.9788	0.9891	0.9869	0.9796	0.9829	0.9793
2	0.9743	0.9880	0.9796	0.9844	0.9801	0.9795	0.9788	0.9846	
3	0.9855	0.9838	0.9784	0.9790	0.9838	0.9860	0.9866		
4	0.9824	0.9782	0.9819	0.9842	0.9864	0.9819			
5	0.9802	0.9822	0.9847	0.9828	0.9804				
7	0.9828	0.9828							