Implied volatility phenomena as markets aversion to risk

11th Financial Risks International Forum, Paris

M. Pontier[1], S. Siyanko[2]

March 26, 2018

[1]Institut Mathématiques de Toulouse, France
[2]University College London

Agenda

Introduction

Motivation

- Technical trading can make assets over or undervalued with respect to their true, intrinsic value defined by the fundamental analysis.
- Market participants may expect a correction in the asset price, and the correction would be sudden and unpredictable.
- Option sellers will see the possibility of price corrections as a risk and price it as such.
- We hypothesize that the premium for the risk of price corrections is indeed included in option prices, and it manifests itself through market phenomena called volatility skew and volatility smile.

Objective

The objective is to introduce an algorithm for pricing derivatives that can explicitly price and measure the risk of price corrections as assumed by the market.

Models with mean-reversion and jumps

- S. Heston model [4]. Mean-reverting process but no jumps. Mean-reversion is motivated by local behavior of the stock (volatility clustering and auto-correlation).
- Geman H. and Roncoroni A. model [3] describing the structure of electricity prices. Jumps direction (but not size) depends on price thresholds which leads to mean-reversion.
- Bellamy N. and Jeanblanc M. model [1]. A wide class of processes driven by a mixed diffusion. It has technical constraints to ensure that solutions are of a certain type.
- Merton's jump-to-ruin model [5] for pricing default risk. This is a special case of our model (when the fundamental value is zero).

Two models for the stock price

Model I. Geometric Brownian motion with Poisson jumps.

Let $\{\Omega, \mathcal{A}, \mathbb{F} = (\mathcal{F}_t, 0 \leq t \leq T), P\}$ be a filtered probability space on which we define a pair of a standard Brownian motion W and a Poisson process N with constant intensity λ. Our stock price dynamics is:

$$dS_t = \mu S_{t-} dt + \sigma S_{t-} dW_t + (\overline{S}_0 e^{\mu t} - S_{t-}) dN_t,$$
$$S_0 \in L^1(\Omega, \mathcal{F}_0, P),\ 0 \leq t \leq \text{ maturity } T,$$

where S_0, W, N are independent, and W, N are $\mathbb{F}$-adapted. Here, we have stock price S_t, its drift rate μ, volatility σ, and time t. In addition, we have the fundamental value of the stock $\overline{S}_0 e^{\mu t} = \overline{S}_t$ with constant $\overline{S}_0$.

Remark The SDE is solvable and admits the unique positive solution as standard geometric Brownian motion evolving from its initial value at time zero or from its fundamental value at jump time T_{N_t}.

Model I. Risk-neutral valuation.

Proposition 2.1 There exists a risk-neutral probability measure:

$$Q = L_T P, \text{ where } L_T = \rho_t^W \rho_t^N,$$

$$d\rho_t^W = \rho_t^W \psi_t dW_t, \quad d\rho_t^N = \rho_{t-}^N \gamma_t (dN_t - \lambda dt).$$

So this market is viable, there is no arbitrage opportunity. We propose:

$$1 + \gamma_t = \inf(S_{t-}, c), \ \psi_t = \frac{r - \mu}{\sigma} + (1 - \frac{\overline{S}_0 e^{\mu t}}{S_{t-}}) \frac{\lambda \inf(S_{t-}, c)}{\sigma}$$

for some constant $c > 0$, and under Q we have:

$$dS_t = rS_{t-}dt + \sigma S_{t-} d\tilde{W}_t + (\overline{S}_0 e^{\mu t} - S_{t-}) d\tilde{M}_t, \ S_0, \ t \leq \text{ maturity } T,$$

$$\tilde{W}_t = W_t - \int_0^t \psi_s ds, \ \tilde{M}_t = N_t - \int_0^t \lambda(1 + \gamma_s) ds.$$

Remark Under Q, $\lambda_t = \lambda \inf(S_{t-}, c)$ and tends to zero when $S_{t-} \to 0$.

Model II. The Model.

Let $\{\Omega, \mathcal{A}, \mathbb{F} = (\mathcal{F}_t, 0 \leq t \leq T), P, Q\}$ be a filtered probability space, filtration $\mathbb{F}$ generated by a standard Brownian motion W and a Poisson process N with constant intensity λ and independent of W. Two equivalent probability measures P and Q. Our stock price process is:

$$dS_t = \mu S_{t-} dt + \sigma S_{t-} d\tilde{W}_t + (\overline{S}_0 e^{\mu t} - S_{t-})(dN_t - \lambda dt),$$
$$S_0 \in \mathbb{R}^+,\ 0 \leq t \leq \text{ maturity } T$$

under P, and

$$dS_t = r S_{t-} dt + \sigma S_{t-} dW_t + (\overline{S}_0 e^{\mu t} - S_{t-})(dN_t - \lambda dt) \qquad (1)$$

under Q, where $dW_t = d\tilde{W}_t + \frac{\mu - r}{\sigma} dt$, $Q = L_T P$, $dL_t = -L_t \frac{\mu - r}{\sigma} d\tilde{W}_t$. We only change the Brownian motion part, and under both P and Q the intensity λ is the same. There exists a risk-neutral probability measure, namely Q, this market is viable, we have AOA.

Model II. Solution to the SDE.

Proposition 2.2 The unique solution to (1) on the event $\{T_{N_t} = 0\}$ is

$$S_t = e^{(r+\lambda-\frac{\sigma^2}{2})t+\sigma W_t}\left(S_0 - \lambda\int_0^t \overline{S}_0 e^{\mu s} e^{-(r+\lambda-\frac{\sigma^2}{2})s-\sigma W_s} ds\right)$$

and similarly, on the event $\{T_{N_t} > 0\}$, we have:

$$S_t = e^{(r+\lambda-\frac{\sigma^2}{2})(t-T_{N_t})+\sigma(W_t-W_{T_{N_t}})}x$$

$$x\left(\overline{S}_0 e^{\mu T_{N_t}} - \lambda\int_{T_{N_t}}^t \overline{S}_0 e^{\mu s} e^{-(r+\lambda-\frac{\sigma^2}{2})(s-T_{N_t})-\sigma(W_s-W_{T_{N_t}})} ds\right).$$

Note that S_t may take negative values, e.g. after time

$$\tau := \inf\{t \geq 0 : \quad \lambda\int_0^t \overline{S}_0 e^{\mu s} e^{-(r+\lambda-\frac{\sigma^2}{2})s-\sigma W_s} ds = S_0\}, \tau < T_1.$$

Remark Model I can not allow constant λ under Q because $Q \sim P$.

Pricing PDE

Pricing PDE. Formal introduction from Model II.

We can invoke the fundamental theorem of asset pricing and write for a call option:

$$C_t = e^{-r(T-t)} E_Q[(S_T - K)^+ / \mathcal{F}_t].$$

Since S is a Markov process, there exists a regular function C such that

$$C_t = C(t, S_t).$$

Proposition 3.1 Assuming the function C defined on $[0, T] \times \mathbb{R}$ is $C^{1,2}$, it is solution to the partial derivatives equation

$$\frac{\partial C}{\partial t}(t,s) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 C}{\partial s^2}(t,s) = r[C(t,s) - s\frac{\partial C}{\partial s}(t,s)] -$$
$$-\lambda[(C(t, \overline{S}_0 e^{\mu t}) - C(t,s)) - \frac{\partial C}{\partial s}(t,s)(\overline{S}_0 e^{\mu t} - s)],$$
$$C(T,s) = (s-K)^+.$$

Remark We have an equation of an integro-differential type.

Pricing PDE. Informal introduction from Model I.

We construct a self-financing portfolio with money market account B_t:

$$V_t = \phi_t S_t + \psi_t B_t$$

and some predictable processes ϕ and ψ. We then try to replicate the option over dt and realize that we can do delta hedge to equate Brownian motion coefficients on both sides but we can not equate coefficients of Poisson process meaning that we can not replicate the option's price process exactly. Best we can do is to replicate the payoff "on average":

$$V_t = C(t, S_t),\ E[C(T, S_T)/\mathcal{F}_t] = E[V_T/\mathcal{F}_t].$$

The above is true for martingales. So, we need to identify and equate finite variation parts of semi-martingales which leads to the same PDE.

Remark The stock price is strictly positive which hints at a possibility to restrict PDE to $s \in R^+ - \{0\}$.

Pricing PDE. Simplified.

Proposition 4.1 We can simplify Pricing PDE further by introducing an auxiliary function u on $[0, T] \times \mathbb{R}^+$ satisfying:

$$C(t,s) = u(t,s)e^{-(r+\lambda)(T-t)} + \lambda e^{-r(T-t)} \int_t^T e^{-\lambda(T-v)} u(v, \overline{S}_0 e^{\mu v}) dv.$$

It makes it equivalent to the following PDE with u being a solution of it:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} = -s(r+\lambda)\frac{\partial u}{\partial s} + \lambda \overline{S}_0 e^{\mu t} \frac{\partial u}{\partial s}, \quad u(T,s) = (s-K)^+.$$

Proposition 4.3 The boundary conditions for the function u are:

$$u(t,+\infty) \quad \approx \quad se^{(r+\lambda)(T-t)} - K - \frac{\lambda \overline{S}_0 e^{\mu t}[e^{\mu(T-t)} - e^{(r+\lambda)(T-t)}]}{\mu - r - \lambda},$$

$$u(t,s) = \partial_s u(t,s) = 0 \text{ for } s = 0.$$

Remark We have a similar PDE in the problem for pricing Asian options.

Pricing PDE. Solving it numerically.

The solution of a floating-strike put Asian PDE can be reduced to the solution of the following PDE (cf. S. Siyanko [6]):

$$\partial_\tau \psi(\xi, \tau, \rho, T, \lambda) = \frac{1}{2}(\xi - \frac{1}{\rho T}(1 - e^{-\rho\tau}))^2 \partial_{\xi\xi}\psi(\xi, \tau, \rho, T, \lambda) \; ;$$

$$\psi(\xi, 0, \rho, T, \lambda) = (\xi - \lambda)^+$$

with some real constants ξ, $\tau \geq 0$, ρ, $T > 0$, and $\lambda > 0$. We can reduce our Pricing PDE to the same PDE through a straightforward change of variables (we also fix time t, $\tau = T - t$, $S_t = s$, and $\bar{s} = \overline{S}_t$):

$$u(t, s) = \psi\left[\zeta\left(s, \bar{s}, \mu, \lambda, \tau, \sigma, K, r\right), \sigma^2\tau, \frac{\mu - r - \lambda}{\sigma^2}, -\frac{\sigma^2}{\lambda\bar{s}e^{\mu\tau}}, K\right],$$

where ζ is defined as a natural *Log* function:

$$\zeta\left(s, \bar{s}, \mu, \lambda, \tau, \sigma, K, r\right) = Log\left[\frac{s}{K}e^{(r+\lambda)\tau} - \frac{\lambda\bar{s}e^{\mu\tau}\left(1 - e^{-(\mu - r - \lambda)\tau}\right)}{K(\mu - r - \lambda)}\right] + \frac{1}{2}\sigma^2\tau.$$

Remark Log function is similar to the one in the Black-Scholes formula.

Pricing algorithms

Pricing algorithms. Matched asymptotic expansion.

We can apply β algorithm from S. Siyanko [6] (extension of work by J. Dewynne and W. Shaw [2]) and expand ψ in asymptotic series:

$$\psi(\zeta,\tau,\rho,L,K) = e^{\zeta-\frac{\tau}{2}} K\Phi_G\left[\frac{\zeta}{\sqrt{\tau}}\right] - K\Phi_G\left[\frac{\zeta-\tau}{\sqrt{\tau}}\right] + \epsilon(\zeta,\tau,\rho,L,K),$$

where Φ_G denotes the cumulative normal distribution function, and ϵ is an asymptotic expansion of the remainder:

$$\epsilon(\zeta,\tau,\rho,L,K) = \frac{1}{\sqrt{2\pi\tau}}\left(\frac{\tau}{L}\right) e^{-\frac{\zeta^2}{2\tau}} \sum_{n=1}^{n_\beta}\sum_{i=0}^{n_\xi}\sum_{j=0}^{n_\tau}\sum_{k=0}^{j-1} \frac{b_{n,i,j,k}\zeta^i\tau^j\rho^k}{n!(KL)^{n-1}},$$

where $\{b_{n,i,j,k}\}$ are real-valued constants.

Remark The pricing algorithm is an adjusted Black-Scholes formula plus a remainder calculated with finite sums.

Pricintg algorithms. Performance vs. finite difference method.

We have benchmarked performance of the algorithm to the finite difference method and found that it performs well withing ranges of parameters used for calibration.

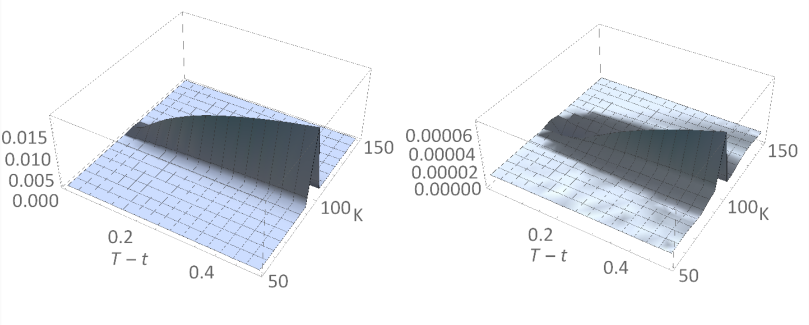

Figure 1: Finite difference vs. the expansion algorithm: difference for $u(\tau, s)$ on left hand and option's Delta on right hand as functions of $\tau = T - t$ and K; $\sigma = 0.2$, $\mu = 4.125\%$, $r = 0.15\%$, $\lambda = 0.25$, $\bar{s} = 100$, $s = 100$.

Numeric results

Numeric results. Calibration strategy.

- One month worth of American call/put options for ten randomly selected stocks traded on NASDAQ.
- The implied volatility shape is not stable over time meaning that model's parameters are not constant, we assume them to be piecewise constant. Hence, we had t fixed and calibrated to one implied volatility shape only.
- Model's outputs have low sensitivity to μ, and it makes sense to set it to a constant.
- Calibration results could be sensitive to $\bar{s}$ from which the calibration process starts. We have therefore calibrated with different starting points (50%, 100% and 150% of the stock price) and selected best optimization results.

Numeric results. Performance vs. Heston model.

We found that, from the goodness of fit point of view, models perform similarly, and errors are strongly correlated. We can obtain a good linear regression model fit ($R^2 = 0.50$), and the correlation coefficient is close to one (0.78).

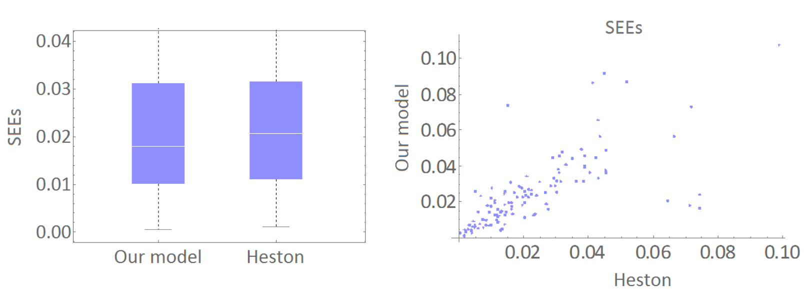

Figure 2: Standard Estimation Errors (SEEs) for our model vs. Heston model, box and scatter plots for SEEs.

Numeric results. Volatility σ vs. implied volatilities.

We can see that for all stocks the median is slightly below the median of implied volatilities and there is a reduction in the data range for σ.

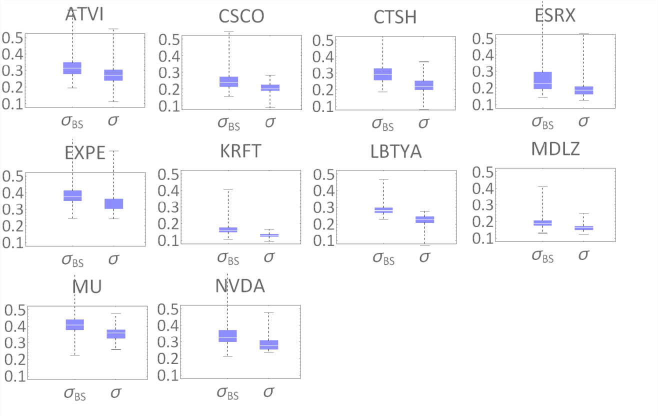

Figure 3: Box plots for implied volatilities (σ_{BS}) left hand vs. volatilities from our model (σ) right hand.

Numeric results. Fundamental value of the stock $\bar{s}$ and λ.

There are three distinct clusters for the fundamental value of the stock: 1) below 10% (skew, default is expected); 2) in the range of $90 - 100\%$ (smile, the stock is perceived to be priced on value); 3) above 150% (skew, an extreme upward correction is expected). λ is concentrated around smaller values, in fact, its median is around 0.5.

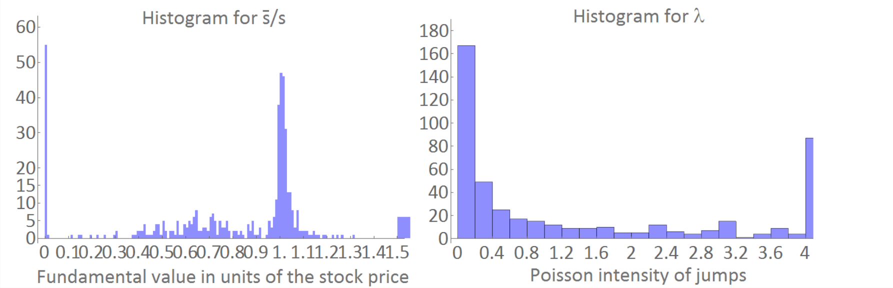

Figure 4: Histograms for $\bar{s}/s$ and λ.

Numeric results. Fundamental value of the stock $\bar{s}$ and λ.

$\bar{s}/s$ and λ are not independent. When stock is perceived as priced on value, we get a spike in the intensity of jumps. High λ and large $|\bar{s}/s - 1|$ would lead to larger perceived profits for technical traders thus driving higher demand and reduce $|\bar{s}/s - 1|$. When λ is small, markets can afford higher misalignment between the stock price and it's fundamental value.

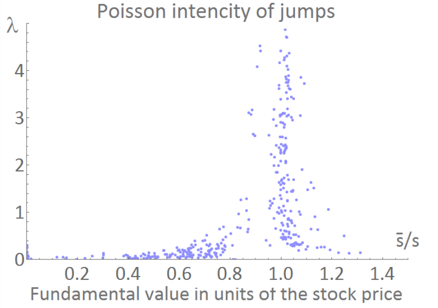

Figure 5: λ versus $\bar{s}/s$.

Numeric results. Summary.

- There is no evidence to suggest that our model performs better or worst compare to Heston model.
- We can often estimate its parameters just by looking at an implied volatility shape, e.g. if we have a smile, it should be centered around $\bar{s}$, and we should expect σ to be slightly below its minimum.
- Since we have calibrated to low values of λ, it may not be efficient to explain the implied volatility phenomenon through local behavior of the stock. Fundamentally, it is a response to the fear of price corrections.

Practical applications

Practical applications

- Pricing derivatives. However, estimating λ from a possibility of future events may not align with the business strategy. One needs to derive it from historic prices in order to stay competitive.
- Measuring the price of risk implied by the market and, if excessive risks are being taken, regulating these markets.
- If markets are pricing derivatives within the default region we can compare λ to the λ implied by credit derivatives markets which may lead to more efficient risk management strategies or even a possibility of an arbitrage.
- We can do technical trading by betting on the stock price moving towards its fundamental value. However, we did not find any evidence to suggest that this strategy would be profitable.

Conclusion

- We have introduced a derivatives pricing model that interprets implied volatility smile and skew as market's aversion to the risk of price corrections. The model takes three new parameters ($\bar{s}$, λ, μ) which can be interpreted and estimated independently of the model.
- The model seems to perform no differently to Heston model but the difference is that we can often estimate its parameters just by looking at an implied volatility shape.
- Practical applications go beyond the traditional derivatives pricing role. The model can help to assess if markets are over or underpricing risks related to price corrections in the underlying.
- While the mathematical community talks about risk-neutral pricing, in practice, there are some real sources of risk, and the possibility of price corrections is one of them.

Acknowledgements

We would like to thank Prof. William Shaw for his guidance and support. The constrictive feedback he gave during our meetings and presentations was invaluable in taking this work to its logical conclusion. We would definitely like to acknowledge contribution from the examiners during MPhil-PhD transfer (Prof. Lane Hughston) and PhD viva (Prof. Sam Howison, Dr. Andrea Macrina). We have also to thank Pr Dominique Lépingle for his valuable help, and the ones who listen to us in Séminaire Bachelier, helping us with their remarks, comments and questions.

Questions?

References i

- N. Bellamy and M. Jeanblanc, *Incompleteness of markets driven by a mixed diffusion*, Finance and Stochastics, 4(2) (2000), pp. 209–222.
- J. Dewynne and W. Shaw, *Differential equations and asymptotic solutions for arithmetic Asian option: "Black-Scholes formulae" for Asian rate calls*, Euro Jnl of Applied Mathematics, (2008).
- H. Geman and A. Roncoroni, *Understanding the fine structure of electricity prices*, The Journal of Business, 79(3) (2006), pp. 1225–1261.
- S. Heston, *A closed-form solution for options with stochastic volatility with applications to bond and currency options*, Review of financial studies, 6(2) (1993), pp. 327–343.

References ii

R. Merton, *On the pricing of corporate debt: The risk structure of interest rates*, The Journal of Finance, 29(2) (1974), pp. 449–470.

S. Siyanko, *Essentially exact asymptotic solutions for Asian derivatives*, European Journal of Applied Mathematics, 23(03) (2012), pp. 395–415.