Estimation risk for the VaR of portfolios driven by semi-parametric multivariate models

Christian Francq Jean-Michel Zakoïan

CREST and University of Lille, France

11th Financial Risks International Forum

26-27 March 2018

Risk of portfolios

- A large strand of the recent literature on quantitative risk management has been concerned with risk aggregation.
- For a vector of one-period P&L random variables $\boldsymbol{y} = (y_1, \dots, y_m)'$, risk aggregation concerns the risk implied by an aggregate financial position defined as a real-valued function of $\boldsymbol{y}$.
- Under the terms of Basel II, banks often measure risk of an aggregate position through the Value-at-Risk (VaR) of $\boldsymbol{a}'\boldsymbol{y} = a_1 y_1 + \cdots + a_m y_m$ where the a_i's define the composition of a portfolio.
- Exact calculation of the risk associated with an aggregate position can represent a difficult task, as it may require **knowledge of the joint distribution** of the components of $\boldsymbol{y}$.

Conditional risk

- In a dynamic framework, one has to evaluate the conditional risk of a portfolio of assets or returns.
- The current regulatory framework for banking supervision (Basel II, III) allows international banks to develop internal models for the calculation of risk. The so-called **advanced approaches** are based on conditional distributions.

Objectives

- **Setup:**
  - the vector of individual returns $\mathbf{y}_t$ follows a general dynamic model.
  - A portfolio of assets with time-varying composition $\mathbf{a}_{t-1}$. The composition can be both time-varying and stochastic: investors may rebalance their portfolios at time t using the information contained in the historical prices.
- **Aims:**
  - Estimate the conditional risk of the portfolio (market risk).
  - Evaluate the accuracy of the estimation (estimation risk): $\Rightarrow$ quantify simultaneously the market and estimation risks.
  - Compare univariate and multivariate approaches.
    - Crystallized portfolios;
    - Optimal (conditional) mean-variance portfolios;
    - Minimal VaR porfolios.

Risk factors

- $\boldsymbol{p}_t = (p_{1t}, \dots, p_{mt})'$ vector of prices of m assets
- $\boldsymbol{y}_t = (y_{1t}, \dots, y_{mt})'$ vector of log-returns, $y_{it} = \log(p_{it}/p_{i,t-1})$
- V_t value of a portfolio composed of $\mu_{i,t-1}$ units of asset i, for $i = 1, \dots, m$:

$$V_t = \sum_{i=1}^{m} \mu_{i,t-1} p_{it}$$

- Self-financing constraint: At time t, the investor may rebalance his portfolio in such a way that

SF $\quad \sum_{i=1}^{m} \mu_{i,t-1} p_{it} = \sum_{i=1}^{m} \mu_{i,t} p_{it}$

Return of the portfolio

Under SF, the return of the portfolio over the period $[t-1,t]$, assuming $V_{t-1} \neq 0$, is

$$\frac{V_t}{V_{t-1}} - 1 = \sum_{i=1}^{m} a_{i,t-1} \exp(y_{it}) - 1 \approx r_t$$

where

$$r_t = \sum_{i=1}^{m} a_{i,t-1} y_{it} = \mathbf{a}'_{t-1} \boldsymbol{y}_t,$$

with

$$a_{i,t-1} = \frac{\mu_{i,t-1} p_{i,t-1}}{\sum_{j=1}^{m} \mu_{j,t-1} p_{j,t-1}}, \quad i = 1,\dots,m,$$

and $\mathbf{a}_{t-1} = (a_{1,t-1},\dots,a_{m,t-1})'$, $\quad \boldsymbol{y}_t = (y_{1t},\dots,y_{mt})'$.

Conditional VaR of the portfolio's return

The *conditional* VaR of the portfolio's return r_t at risk level $\alpha \in (0,1)$ is defined by

$$P_{t-1}\left[r_t < -\text{VaR}_{t-1}^{(\alpha)}(r_t)\right] = \alpha$$

where P_{t-1} denotes the historical distribution conditional on $\{\boldsymbol{p}_u, u < t\}$.

Consequence

Evaluation of the conditional VaR can be achieved by a

- Multivariate approach:
  dynamic model for the vector of risk factors $\boldsymbol{y}_t$
- Univariate approach:
  dynamic model for the portfolio's return r_t

Dynamic model for the vector of log-returns

Multivariate model with GARCH-type errors:

$$\boldsymbol{y}_t = \boldsymbol{m}_t(\boldsymbol{\theta}_0) + \boldsymbol{\epsilon}_t, \qquad \boldsymbol{\epsilon}_t = \boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0)\boldsymbol{\eta}_t$$

where $\boldsymbol{\eta}_t \overset{iid}{\sim} (\mathbf{0}, \boldsymbol{I}_m)$, $\quad \boldsymbol{\theta}_0 \in \mathbb{R}^d$

$$\boldsymbol{m}_t(\boldsymbol{\theta}_0) = \boldsymbol{m}(\boldsymbol{y}_{t-1}, \boldsymbol{y}_{t-2}, \dots, \boldsymbol{\theta}_0), \qquad \boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0) = \boldsymbol{\Sigma}(\boldsymbol{y}_{t-1}, \boldsymbol{y}_{t-2} \dots, \boldsymbol{\theta}_0).$$

Examples of MGARCH

Thus

$$r_t = \mathbf{a}'_{t-1}\boldsymbol{m}_t(\boldsymbol{\theta}_0) + \mathbf{a}'_{t-1}\boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0)\boldsymbol{\eta}_t,$$

and

$$\mathrm{VaR}_{t-1}^{(\alpha)}(r_t) = -\mathbf{a}'_{t-1}\boldsymbol{m}_t(\boldsymbol{\theta}_0) + \mathrm{VaR}_{t-1}^{(\alpha)}\left(\mathbf{a}'_{t-1}\boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0)\boldsymbol{\eta}_t\right).$$

A simplification for elliptic conditional distributions

$$\boldsymbol{\epsilon}_t = \boldsymbol{m}_t(\boldsymbol{\theta}_0) + \boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0)\boldsymbol{\eta}_t, \qquad (\boldsymbol{\eta}_t) \text{ iid } (\mathbf{0}, \boldsymbol{I}_m),$$

Assume that the errors $\boldsymbol{\eta}_t$ have a spherical distribution:

A1: for any non-random vector $\boldsymbol{\lambda} \in \mathbb{R}^m$, $\boldsymbol{\lambda}'\boldsymbol{\eta}_t \overset{d}{=} \|\boldsymbol{\lambda}\|\eta_{1t}$

where $\|\cdot\|$ is the euclidean norm on $\mathbb{R}^m$.

Remark: means that the conditional law of $\boldsymbol{\epsilon}_t$ is elliptic.

Under **A1**

$$\text{VaR}_{t-1}^{(\alpha)}(r_t) = -\mathbf{a}'_{t-1}\boldsymbol{m}_t(\boldsymbol{\theta}_0) + \left\|\mathbf{a}'_{t-1}\boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0)\right\| \text{VaR}^{(\alpha)}(\eta),$$

where $\text{VaR}^{(\alpha)}(\eta)$ is the (marginal) VaR of η_{1t}.

Assumption on the conditional variance model

B1: There exists a continuously differentiable function $\boldsymbol{G} : \mathbb{R}^d \mapsto \mathbb{R}^d$ such that for any $\boldsymbol{\theta} \in \Theta$, any $K > 0$, and any sequence $(\mathbf{x}_i)_i$ on $\mathbb{R}^m$,

$$K\boldsymbol{\Sigma}(\mathbf{x}_1, \mathbf{x}_2, \ldots; \boldsymbol{\theta}) = \boldsymbol{\Sigma}(\mathbf{x}_1, \mathbf{x}_2, \ldots; \boldsymbol{\theta}^*), \quad \text{and}$$
$$m(\mathbf{x}_1, \mathbf{x}_2, \ldots; \boldsymbol{\theta}) = m(\mathbf{x}_1, \mathbf{x}_2, \ldots; \boldsymbol{\theta}^*)$$

where $\boldsymbol{\theta}^* = \boldsymbol{G}(\boldsymbol{\theta}, K)$.

VaR parameter for an elliptic conditional distribution

At the risk level $\alpha \in (0, 0.5)$, the conditional VaR of the portfolio's return is

$$\begin{aligned}
\text{VaR}_{t-1}^{(\alpha)}(r_t) &= -\mathbf{a}_{t-1}' \boldsymbol{m}_t(\boldsymbol{\theta}_0) + \text{VaR}_{t-1}^{(\alpha)}\left(\mathbf{a}_{t-1}' \boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0) \boldsymbol{\eta}_t\right) \\
&= -\mathbf{a}_{t-1}' \boldsymbol{m}_t(\boldsymbol{\theta}_0) + \left\| \mathbf{a}_{t-1}' \boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0) \right\| \text{VaR}^{(\alpha)}(\eta) \\
&= -\mathbf{a}_{t-1}' \boldsymbol{m}_t(\boldsymbol{\theta}_0^*) + \| \mathbf{a}_{t-1}' \boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0^*) \|,
\end{aligned}$$

where, under **B1**,

$$\boldsymbol{\theta}_0^* = G\left(\boldsymbol{\theta}_0, \text{VaR}^{(\alpha)}(\eta)\right).$$

The parameter $\boldsymbol{\theta}_0^*$ can be called conditional VaR parameter.

Remark: The conditional VaR parameter

- does not depend on the portfolio composition
- summarizes the risk at a given level

1. General framework
2. Estimating the conditional VaR
    - Multivariate estimation under ellipticity
    - Relaxing the ellipticity assumption
    - Univariate approaches
3. Numerical comparison of the different VaR estimators

Estimating the conditional VaR parameter

- Observations: $\boldsymbol{y}_1,\dots,\boldsymbol{y}_n$ (+ initial values $\tilde{\boldsymbol{y}}_0,\tilde{\boldsymbol{y}}_{-1},\dots$).
- $\widehat{\boldsymbol{\theta}}_n$: estimator of $\boldsymbol{\theta}_0$
- $\tilde{\boldsymbol{m}}_t(\boldsymbol{\theta}) = \boldsymbol{m}(\boldsymbol{y}_{t-1},\dots,\boldsymbol{y}_1,\tilde{\boldsymbol{y}}_0,\tilde{\boldsymbol{y}}_{-1},\dots,\boldsymbol{\theta})$
  $\tilde{\boldsymbol{\Sigma}}_t(\boldsymbol{\theta}) = \boldsymbol{\Sigma}(\boldsymbol{y}_{t-1},\dots,\boldsymbol{y}_1,\tilde{\boldsymbol{y}}_0,\tilde{\boldsymbol{y}}_{-1},\dots,\boldsymbol{\theta})$
- Residuals: $\widehat{\boldsymbol{\eta}}_t = \tilde{\boldsymbol{\Sigma}}_t^{-1}(\widehat{\boldsymbol{\theta}}_n)\{\boldsymbol{y}_t - \tilde{\boldsymbol{m}}_t(\widehat{\boldsymbol{\theta}}_n)\}) = (\widehat{\eta}_{1t},\dots,\widehat{\eta}_{mt})'$.

Under the sphericity assumption,

$$\widehat{\mathrm{VaR}}^{(\alpha)}_{S,t-1}(r) = -\mathbf{a}'_{t-1}\tilde{\boldsymbol{m}}_t(\widehat{\boldsymbol{\theta}}_n) + \|\mathbf{a}'_{t-1}\tilde{\boldsymbol{\Sigma}}_t(\widehat{\boldsymbol{\theta}}_n)\|\widehat{\mathrm{VaR}}^{(\alpha)}_n(\eta)$$

where $\widehat{\mathrm{VaR}}^{(\alpha)}_n(\eta) = \xi_{n,1-2\alpha}$
is the $(1-2\alpha)$-quantile of $\{|\widehat{\eta}_{it}|, 1 \le i \le m, 1 \le t \le n\}$.

Assumptions

A2: $(\boldsymbol{y}_t)$ is a strictly stationary and nonanticipative solution.

A3: We have $\widehat{\boldsymbol{\theta}}_n \to \boldsymbol{\theta}_0$, a.s. and the following expansion

$$\sqrt{n}\left(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0\right) \overset{o_P(1)}{=} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \boldsymbol{\Delta}_{t-1} \boldsymbol{V}(\boldsymbol{\eta}_t),$$

where $\boldsymbol{\Delta}_{t-1} \in \mathscr{F}_{t-1}$, $\boldsymbol{V} : \mathbb{R}^m \mapsto \mathbb{R}^K$ for some $K \geq 1$, $E\boldsymbol{V}(\boldsymbol{\eta}_t) = 0$, var$\{\boldsymbol{V}(\boldsymbol{\eta}_t)\} = \boldsymbol{\Upsilon}$ is nonsingular and $E\boldsymbol{\Delta}_t = \boldsymbol{\Lambda}$ is full row rank.

A4: The functions $\boldsymbol{\theta} \mapsto \boldsymbol{m}(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots; \boldsymbol{\theta})$ and $\boldsymbol{\theta} \mapsto \boldsymbol{\Sigma}(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots; \boldsymbol{\theta})$ are $\mathscr{C}^1$.

A5: $|\eta_{1t}|$ has a density f which is continuous and strictly positive in a neighborhood of $\xi_{1-2\alpha}$ (the $(1-2\alpha)$-quantile of $|\eta_{1t}|$).

Asymptotic distribution

Asymptotic normality

$$\sqrt{n}\begin{pmatrix} \hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}_0 \\ \xi_{n,1-2\alpha} - \xi_{1-2\alpha} \end{pmatrix} \xrightarrow{\mathscr{L}} \mathscr{N}\left(\mathbf{0}, \boldsymbol{\Xi} := \begin{pmatrix} \boldsymbol{\Psi} & \boldsymbol{\Xi}_{\boldsymbol{\theta}\xi} \\ \boldsymbol{\Xi}'_{\boldsymbol{\theta}\xi} & \zeta_{1-2\alpha} \end{pmatrix}\right),$$

where $\boldsymbol{\Omega}' = E\left[\left\{\text{vec}\left(\boldsymbol{\Sigma}_t^{-1}\right)\right\}'\left\{\frac{\partial}{\partial \boldsymbol{\vartheta}'}\text{vec}\left(\boldsymbol{\Sigma}_t\right)\right\}\right]$, $\boldsymbol{W}_\alpha = \text{Cov}(\boldsymbol{V}(\boldsymbol{\eta}_t), N_t)$, $\gamma_\alpha = \text{var}(N_t)$, with $N_t = \sum_{j=1}^m \mathbf{1}_{\{|\eta_{jt}|<\xi_{1-2\alpha}\}} - 1 + 2\alpha$, and

$$\boldsymbol{\Xi}_{\boldsymbol{\theta}\xi} = \frac{-1}{m}\left\{\xi_{1-2\alpha}\boldsymbol{\Psi}\boldsymbol{\Omega} + \frac{1}{f(\xi_{1-2\alpha})}\boldsymbol{\Lambda}\boldsymbol{W}_\alpha\right\}, \quad \boldsymbol{\Psi} = E(\boldsymbol{\Delta}_t\boldsymbol{\Upsilon}\boldsymbol{\Delta}_t')$$

$$\zeta_{1-2\alpha} = \frac{1}{m^2}\left\{\xi_{1-2\alpha}^2\boldsymbol{\Omega}'\boldsymbol{\Psi}\boldsymbol{\Omega} + \frac{2\xi_{1-2\alpha}}{f(\xi_{1-2\alpha})}\boldsymbol{\Omega}'\boldsymbol{\Lambda}\boldsymbol{W}_\alpha + \frac{\gamma_\alpha}{f^2(\xi_{1-2\alpha})}\right\}.$$

Estimation of the asymptotic variance

- Most quantities involved in the asymptotic covariance matrix Ξ can be estimated by empirical means.
- The estimation of

$$\mathbf{\Omega}' = E\left[\left\{\text{vec}\left(\mathbf{\Sigma}_t^{-1}\right)\right\}'\left\{\frac{\partial}{\partial\boldsymbol{\vartheta}'}\text{vec}\left(\mathbf{\Sigma}_t\right)\right\}\right]$$

can be delicate due to the presence of the derivatives of $\mathbf{\Sigma}_t$.

Example: linear SRE on the derivatives of H_t

Asymptotic normality of the VaR-parameter estimator

$$\text{VaR-parameter:} \quad \boldsymbol{\theta}_0^* = G\left(\boldsymbol{\theta}_0, \text{VaR}^{(\alpha)}(\eta)\right)$$

A simple application of the delta method gives the asymptotic distribution of the estimator

$$\widehat{\boldsymbol{\theta}}_n^* = G\left\{\widehat{\boldsymbol{\theta}}_n, \widehat{\text{VaR}}_n^{(\alpha)}(\eta)\right\}.$$

VaR parameter

$$\sqrt{n}\left(\widehat{\boldsymbol{\theta}}_n^* - \boldsymbol{\theta}_0^*\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(\mathbf{0}, \Xi^* := \dot{\boldsymbol{G}}\Xi\dot{\boldsymbol{G}}'\right)$$

with

$$\dot{\boldsymbol{G}} = \left[\frac{\partial G(\boldsymbol{\theta}, \xi)}{\partial(\boldsymbol{\theta}', \xi)}\right]_{(\boldsymbol{\theta}_0, \xi_{1-2\alpha})}.$$

Evaluation of the estimation risk

$$\widehat{\text{VaR}}^{(\alpha)}_{S,t-1}(r) = -\mathbf{a}'_{t-1}\widetilde{\boldsymbol{m}}_t(\widehat{\boldsymbol{\theta}}_n) + \|\mathbf{a}'_{t-1}\widetilde{\boldsymbol{\Sigma}}_t(\widehat{\boldsymbol{\theta}}_n)\|\widehat{\text{VaR}}^{(\alpha)}_n(\eta)$$

An asymptotic $(1-\alpha_0)$% confidence interval for $\mathrm{VaR}_t(\alpha)$ has bounds given by

$$\widehat{\text{VaR}}^{(\alpha)}_{S,t-1}(r_t) \pm \frac{1}{\sqrt{n}}\Phi^{-1}_{1-\alpha_0/2}\left\{\boldsymbol{\delta}'_{t-1}\widehat{\boldsymbol{\Xi}}\boldsymbol{\delta}_{t-1}\right\}^{1/2},$$

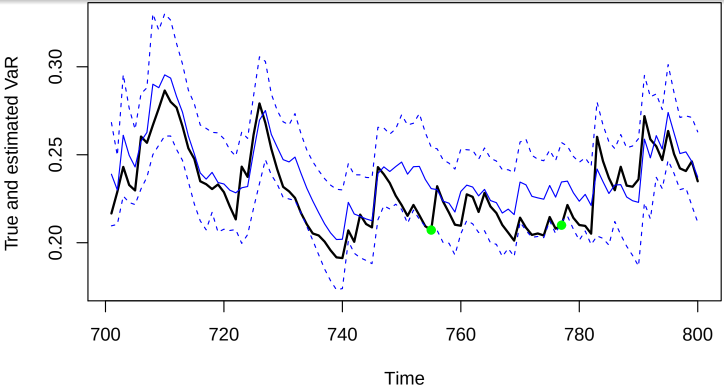
where

$$\boldsymbol{\delta}'_{t-1} = \left[\mathbf{a}'_{t-1}\frac{\partial\widetilde{\boldsymbol{m}}(\widehat{\boldsymbol{\theta}}_n)}{\partial\boldsymbol{\theta}'} + \frac{(\mathbf{a}_{t-1}\otimes\mathbf{a}_{t-1})'}{2\|\mathbf{a}'_{t-1}\widetilde{\boldsymbol{\Sigma}}_t(\widehat{\boldsymbol{\theta}}_n)\|}\frac{\partial \text{vec}\widetilde{H}_t(\widehat{\boldsymbol{\theta}}_n)}{\partial\boldsymbol{\theta}'} \qquad \|\mathbf{a}'_{t-1}\widetilde{\boldsymbol{\Sigma}}_t(\widehat{\boldsymbol{\theta}}_n)\|\right],$$

with $\widetilde{\boldsymbol{H}}_t(\cdot) = \widetilde{\boldsymbol{\Sigma}}_t(\cdot)\widetilde{\boldsymbol{\Sigma}}'_t(\cdot)$.

Remark: The statistical estimation risk α_0 is not related to the financial risk α.

Accuracy intervals for the estimated conditional VaR



1%-VaR (**true** in full black line, estimated in full blue line) and estimated 95%-confidence intervals (dotted blue line) on a simulation of a fixed portfolio of a bivariate BEKK (700 values for the estimation of the VaR parameter).

CI's based on a conditional resampling scheme

To circumvent difficulties due to the estimation of the asymptotic variance Ξ, a bootstrap procedure can be developed.

Relies on the property:
$\|\boldsymbol{\eta}_t\|$ and $\boldsymbol{\eta}_t/\|\boldsymbol{\eta}_t\|$ are independent, the latter being uniformly distributed over $\mathscr{S}^{m-1}$.

Resampling scheme

1. Compute $\widehat{\boldsymbol{\theta}}_n = \widehat{\boldsymbol{\theta}}_n(\boldsymbol{y}_1, \dots, \boldsymbol{y}_n)$, $\widetilde{\boldsymbol{\eta}}_t$, and $\widehat{\text{VaR}}^{(\alpha)}_{S,t-1}(r_t) =: \widehat{\text{VaR}}(r_t)$.
2. Generate independently, for $u = 1, \dots, n$,
   - vectors $\boldsymbol{s}^*_u$ uniformly distributed over $\mathscr{S}^{m-1}$.
   - vectors $\widetilde{\boldsymbol{U}}^*_u$, uniformly distributed on $(\widetilde{\boldsymbol{U}}_1, \dots, \widetilde{\boldsymbol{U}}_n)$. where $\widetilde{\boldsymbol{U}}_u = \boldsymbol{S}_u^{-1/2}(\widetilde{\boldsymbol{\eta}}_u - \widetilde{\boldsymbol{\eta}})$,

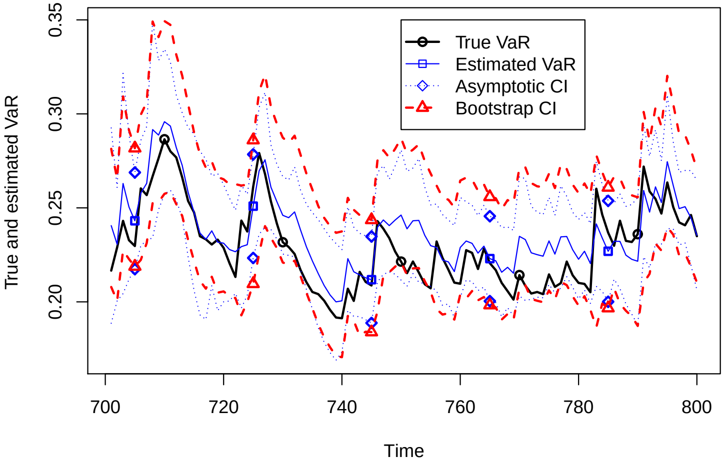
   Compute $\boldsymbol{\eta}^*_u = \|\widetilde{\boldsymbol{U}}^*_u\| \boldsymbol{s}^*_u$ and let

   $$\boldsymbol{y}^*_u = \widetilde{\boldsymbol{m}}^*_u(\widehat{\boldsymbol{\theta}}_n) + \widetilde{\boldsymbol{\Sigma}}^*_u(\widehat{\boldsymbol{\theta}}_n)\boldsymbol{\eta}^*_u$$

   (where $\widetilde{\boldsymbol{m}}^*_u(\widehat{\boldsymbol{\theta}}_n) = \boldsymbol{m}(\boldsymbol{y}^*_{t-1}, \dots, \boldsymbol{y}^*_1, \widetilde{\boldsymbol{y}}_0, \widetilde{\boldsymbol{y}}_{-1}, \dots, \widehat{\boldsymbol{\theta}}_n)$, $\quad \widetilde{\boldsymbol{\Sigma}}^*_u(\widehat{\boldsymbol{\theta}}_n) = ..$)
3. Compute $\widehat{\boldsymbol{\theta}}^*_n = \widehat{\boldsymbol{\theta}}_n(\boldsymbol{y}^*_1, \dots, \boldsymbol{y}^*_n)$, $\widetilde{\boldsymbol{\eta}}^*_u = \widetilde{\boldsymbol{\Sigma}}_u^{-1}(\widehat{\boldsymbol{\theta}}^*_n)\{\boldsymbol{y}^*_u - \widetilde{\boldsymbol{m}}_u(\widehat{\boldsymbol{\theta}}^*_n)\}$ (where $\widetilde{\boldsymbol{m}}_u(\widehat{\boldsymbol{\theta}}^*_n) = \boldsymbol{m}(\boldsymbol{y}_{t-1}, \dots, \boldsymbol{y}_1, \widetilde{\boldsymbol{y}}_0, \widetilde{\boldsymbol{y}}_{-1}, \dots, \widehat{\boldsymbol{\theta}}^*_n)$, $\quad \widetilde{\boldsymbol{\Sigma}}_u(\widehat{\boldsymbol{\theta}}_n) = \dots$) and

   $$\widehat{\text{VaR}}^*(r_t) = -\mathbf{a}'_{t-1}\widetilde{\boldsymbol{m}}_t(\widehat{\boldsymbol{\theta}}^*_n) + \|\mathbf{a}'_{t-1}\widetilde{\boldsymbol{\Sigma}}_t(\widehat{\boldsymbol{\theta}}^*_n)\|\xi^*_{n,1-2\alpha}.$$
4. Repeat B times Steps 1-3, resulting in $\widehat{\text{VaR}}^*_1(r_t), \dots \widehat{\text{VaR}}^*_B(r_t)$.

Accuracy intervals for the estimated conditional VaR



Confidence Intervals based on asymptotic results vs bootstrap.

Filtered Historical Simulation (FHS) approach

Barone-Adesi et al. (J. of Future Markets, 1999), Mancini and Trojani (JFE, 2011)

Relies on

i) interpreting the conditional VaR as the α-quantile of a linear combination (depending on t) of the components of $\boldsymbol{\eta}_t$:

$$\text{VaR}_{t-1}^{(\alpha)}(r_t) = \text{VaR}_{t-1}^{(\alpha)}\left\{b_t(\boldsymbol{\theta}_0) + \boldsymbol{c}_t'(\boldsymbol{\theta}_0)\boldsymbol{\eta}_t\right\}$$

where $b_t(\boldsymbol{\theta}) = \mathbf{a}_{t-1}'\boldsymbol{m}_t(\boldsymbol{\theta})$ and $\boldsymbol{c}_t'(\boldsymbol{\theta}) = \mathbf{a}_{t-1}'\boldsymbol{\Sigma}_t(\boldsymbol{\theta})$.

ii) replacing $\boldsymbol{\eta}_t$ by the GARCH residuals $\widehat{\boldsymbol{\eta}}_s$ and computing the empirical α-quantile of the estimated linear combination.

$$\widehat{\text{VaR}}_{FHS,t-1}^{(\alpha)}(r) = -q_\alpha\left(\{b_t(\widehat{\boldsymbol{\theta}}_n) + \boldsymbol{c}_t'(\widehat{\boldsymbol{\theta}}_n)\widehat{\boldsymbol{\eta}}_s, \quad 1 \le s \le n\}\right).$$

Remark: for each value of s, $b_t(\widehat{\boldsymbol{\theta}}_n) + \boldsymbol{c}_t'(\widehat{\boldsymbol{\theta}}_n)\widehat{\boldsymbol{\eta}}_s$ is a simulated value of the return r_t conditional on the past prices.

Notations and assumptions

Let $\boldsymbol{c}:\boldsymbol{\Theta}\mapsto\mathbb{R}^m$ and $b:\boldsymbol{\Theta}\mapsto\mathbb{R}$ be $\mathscr{C}^1$ functions.

$\xi_\alpha(\boldsymbol{\theta})$: α-quantile of $b(\boldsymbol{\theta})+\boldsymbol{c}'(\boldsymbol{\theta})\boldsymbol{\eta}_t(\boldsymbol{\theta})$,

$\xi_{n,\alpha}(\boldsymbol{\theta})$: empirical α-quantile of $\{b(\boldsymbol{\theta})+\boldsymbol{c}'(\boldsymbol{\theta})\boldsymbol{\eta}_t(\boldsymbol{\theta}),1\leq t\leq n\}$.

Suppose $\xi_\alpha(\boldsymbol{\theta}_0)>0$ and $\boldsymbol{c}'(\boldsymbol{\theta}_0)\boldsymbol{\eta}_t$ admits a density f_c which is continuous and strictly positive in a neighborhood of $x_0=-b(\boldsymbol{\theta}_0)+\xi_\alpha(\boldsymbol{\theta}_0)$.

Asymptotic distribution

Estimator of the quantile of a linear combination of η_t

Under the previous assumptions (but without the sphericity assumption **A1**),

$$\sqrt{n}\{\xi_{n,\alpha}(\widehat{\boldsymbol{\theta}}_n)-\xi_\alpha(\boldsymbol{\theta}_0)\} \stackrel{\mathscr{L}}{\to} \mathscr{N}\left(0, \sigma^2 := \boldsymbol{\omega}'\boldsymbol{\Psi}\boldsymbol{\omega} + 2\boldsymbol{\omega}'\boldsymbol{\Lambda}\boldsymbol{A}_\alpha + \frac{\alpha(1-\alpha)}{f_{\boldsymbol{c}}^2(x_0)}\right),$$

where $\boldsymbol{A}_\alpha = \text{Cov}(V(\boldsymbol{\eta}_t), \mathbf{1}_{\{b(\boldsymbol{\theta}_0)-\boldsymbol{c}'(\boldsymbol{\theta}_0)\boldsymbol{\eta}_t<\xi_\alpha(\boldsymbol{\theta}_0)\}})$,

$$\boldsymbol{\omega}' = \left[\boldsymbol{c}'(\boldsymbol{\theta}_0)E(\boldsymbol{C}_t) - \frac{\partial b}{\partial \boldsymbol{\theta}'}(\boldsymbol{\theta}_0) \quad \boldsymbol{d}_\alpha'\left\{(\boldsymbol{c}'(\boldsymbol{\theta}_0)\otimes \boldsymbol{I}_m)E(\boldsymbol{\Omega}_t^*) - \frac{\partial \boldsymbol{c}}{\partial \boldsymbol{\theta}'}(\boldsymbol{\theta}_0)\right\}\right],$$

$\boldsymbol{d}_\alpha = E(\boldsymbol{\eta}_t \mid b(\boldsymbol{\theta}_0)+\boldsymbol{c}'(\boldsymbol{\theta}_0)\boldsymbol{\eta}_t = \xi_\alpha(\boldsymbol{\theta}_0))$,
$\boldsymbol{\Omega}_t^*$ and $\boldsymbol{C}_t$ are matrices involving the derivatives of $\boldsymbol{\Sigma}_t$ and $\boldsymbol{m}_t$.

Two univariate approaches

- Naive approach: estimate a univariate GARCH model on the series of portfolio returns.
  Generally invalid due to the time-varying combination of the individual returns.
- Virtual Historical Simulation (VHS): reconstitute a "virtual portfolio" whose returns are built using the current composition of the portfolio.

Invalidity of the naive univariate approach

- For crystallized portfolios ($\mu_{i,t-1} = \mu_i, \forall i, \forall t$), in general

$$P(\boldsymbol{a}_{t-1} \in \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}) \to 1 \quad \text{as } t \to \infty.$$

  The composition tends to be totally undiversified, but is not always close to the same single-asset composition $\boldsymbol{e}_i$.

  Illustration of the nonstationarity

  In general, the naive method based on a fixed stationary model for r_t will produce poor results.

- For static portfolios ($a_{i,t-1} = a_i$ for all i and t) the non stationarity issue vanishes.

  However, on simulated series, multivariate models outperform univariate models for estimating the VaR's of static portfolios.

Virtual Historical Simulation

Given the current portfolio composition $\boldsymbol{a}_{t-1} = \boldsymbol{x}$, we construct a (stationary) series of virtual returns mimicking the current return

$$r_s^*(\boldsymbol{x}) = \boldsymbol{x}'\boldsymbol{y}_s \qquad s \in \mathbb{Z}.$$

We have a model of the form

$$r_s^*(\boldsymbol{x}) = \mu_s(\boldsymbol{x}) + \sigma_s(\boldsymbol{x})u_s, \qquad E_{s-1}(u_s) = 0, \quad \mathrm{var}_{s-1}(u_s) = 1.$$

The conditional VaR thus satisfies

$$\mathrm{VaR}_{t-1}^{(\alpha)}(r_t) = -\mu_t(\boldsymbol{a}_{t-1}) + \sigma_t(\boldsymbol{a}_{t-1})\mathrm{VaR}_{t-1}^{(\alpha)}(u_t)$$

STEP 1: Compute the virtual returns $r_s^*(\boldsymbol{x})$ for $s = 1, \dots, n$.
STEP 2: Estimate $\mu_s(\boldsymbol{x})$ and $\sigma_s(\boldsymbol{x})$. Let $\hat{u}_s = \{r_s^*(\boldsymbol{x}) - \hat{\mu}_s(\boldsymbol{x})\}/\hat{\sigma}_s(\boldsymbol{x})$.
STEP 3: Compute the α-quantile $\xi_{n,\alpha}^u(\boldsymbol{x})$ of $\{\hat{u}_s, 1 \le s \le n\}$ and let

$$\widehat{\mathrm{VaR}}_{VHS,t-1}^{(\alpha)}(r) = -\hat{\mu}_t(\boldsymbol{x}) - \hat{\sigma}_t(\boldsymbol{x})\xi_{n,\alpha}^u(\boldsymbol{x}).$$

Remarks on Step 2: estimation of a univariate model for the virtual returns

- To obtain asymptotic properties of the procedure, we make parametric assumptions on the univariate model:

$$\sigma_s(\boldsymbol{x};\boldsymbol{\varrho}) = \sigma(r^*_{s-1}(\boldsymbol{x}), r^*_{s-2}(\boldsymbol{x}), \dots; \boldsymbol{\varrho}),$$

- In general, a multivariate GARCH-type model for $\boldsymbol{y}_t$ is not compatible with a univariate GARCH for $r^*_s(\boldsymbol{x}) = \boldsymbol{x}'\boldsymbol{y}_s$.
  - Due to the fact that the conditional distribution of $r^*_s(\boldsymbol{x})$ is not only a function of the past virtual returns.
  - If a GARCH(1,1) is used in Step 2, it will generally be an approximation.
- Under the sphericity assumption **A1**, (u_t) is i.i.d.

1. General framework
2. Estimating the conditional VaR
3. Numerical comparison of the different VaR estimators
   - On dynamic portfolios
   - On portfolios of exchange rates
   - Appendix

Simulation designs

- Different cDCC-GARCH(1,1) models for $m = 2$ assets. Designs
- For the Minimum variance portfolio Illustration

$$r_t^* = \boldsymbol{\epsilon}_t' \boldsymbol{a}_{t-1}^*, \quad \boldsymbol{a}_{t-1}^* = \frac{\boldsymbol{\Sigma}_t^{-2}(\boldsymbol{\theta}_0)\boldsymbol{e}}{\boldsymbol{e}'\boldsymbol{\Sigma}_t^{-2}(\boldsymbol{\theta}_0)\boldsymbol{e}},$$

  the true conditional VaR is explicit under sphericity, and is evaluated by means of simulations otherwise.
- $N = 100$ independent simulations of the cDCC-GARCH(1,1) model.
  - First $n_1 = 1000$ observations: estimation of $\boldsymbol{\theta}_0$ + empirical quantiles of the residuals.
  - Last $n - n_1 = 1000$ simulations: comparison of the theoretical conditional VaR's of the portfolio with the three estimates (spherical, FHS and VHS methods).

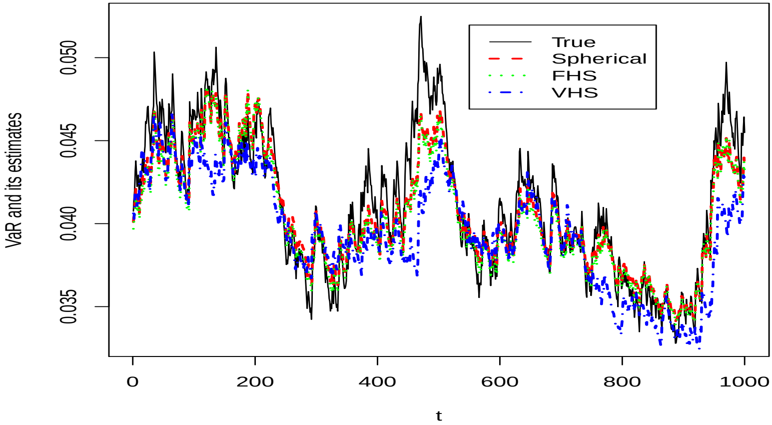
Empirical Relative Efficiency

Table: Relative efficiency of the Spherical method with respect to the FHS method (S/F) and with respect to the VHS method (S/V).

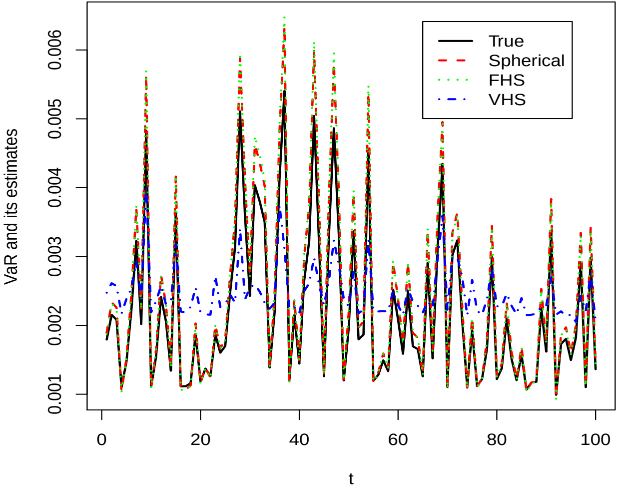
n_1	α		A	B	C	D	E	F	G	H	BEKK
1000	1%	S/F	1.30	1.11	2.35	1.62	1.53	1.51	1.57	1.36	1.41
		S/V	91.6	23.4	303.	79.8	1.93	2.53	4.43	2.23	8.27
	5%	S/F	1.14	1.03	2.07	1.00	1.25	1.08	1.33	1.01	1.13
		S/V	55.4	15.7	267.	82.5	1.75	2.44	4.14	2.01	8.23
			A*	B*	C*	D*	E*	F*	G*	H*	BEKK*
1000	1%	S/F	0.08	0.03	0.02	0.02	0.06	0.03	0.03	0.04	0.05
		S/V	2.20	2.43	2.31	1.67	0.05	0.04	0.07	0.06	0.50
	5%	S/F	0.34	0.19	0.09	0.11	0.30	0.24	0.21	0.29	0.34
		S/V	3.78	6.68	10.2	8.72	0.26	0.35	0.59	0.44	2.65

A-H: Spherical innovations; A*-H*: Non spherical innovations

The two components follow persistent volatility models



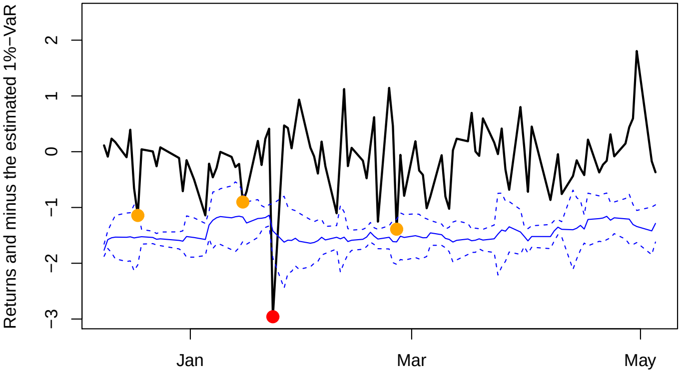
Two very different volatility models for the two components (design A)



Daily returns of exchange rates against the Euro

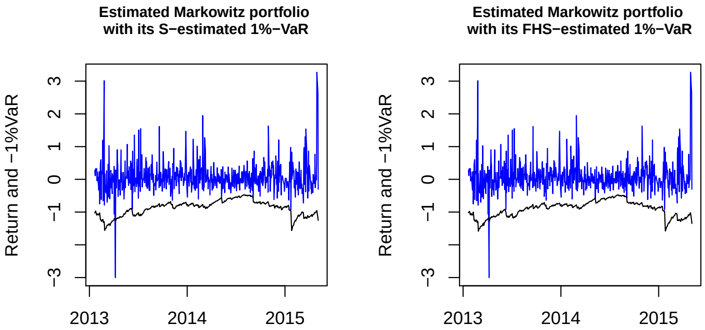
- Canadian Dollar (CAD), Chinese Yuan (CNY), British Pound (GBP), Japanese Yen (JPY) and US Dollar (USD).
- January 14, 2000 to May 5, 2015 ($n = 2582$).
- 2 settings
  - A BEKK model estimated over the whole sample except the last 100 returns. Equally-weighted crystalized portfolio ($\mu_i = 1$ for $i = 1, \dots, 5$). VaR estimates based on sphericity.
  - DCC GARCH(1,1) model on the first 2000 observations with estimated minimum-variance portfolio. Backtesting (unconditional coverage, independence of violations, conditional coverage).

Equally-weighted portfolio of 5 exchange rates



Returns for the period 09/12/2014 to 05/05/2015, estimated 1%- VaR and 95%-confidence interval based on the estimation of a BEKK model.

Minimum-variance portfolio of 5 exchange rates



Returns of estimated minimum-variance portfolios of 5 exchange rates and their estimated VaR's.

Backtests

Christoffersen (2003), Escanciano and Olmo (2010, 2011)

Table: p-values of three backtests for minimum-variance portfolios

Method	α	% of Viol	UC	IND	CC
Spherical	1%	2/582	0.065	0.906	0.182
FHS	1%	2/582	0.065	0.906	0.182
Spherical	5%	20/582	0.067	0.232	0.092
FHS	5%	18/582	0.023	0.283	0.043

Conclusions: univariate approaches

- Not always a good idea to fit a stationary univariate GARCH model on portfolios returns:
  - does not exploit the multivariate dynamics of the risk factors;
  - the naive approach (based on a fixed stationary model) is generally inconsistent when the composition of the portfolio is time-varying;
  - The VHS approach circumvents the non stationarity problem but
    - is generally found inefficient in simulations compared to the multivariate approaches,
    - is not necessarily simpler to implement (GARCH models have to be re-estimated at any date and for any portfolio composition),
    - does not allow to choose optimally the weights of the portfolio.

Conclusions: multivariate approaches

- For both approaches, asymptotic CIs for the conditional VaR can be built.
  $\Rightarrow$ allows to visualize on the same graph both market and estimation risks.
- Exploiting the sphericity simplifies estimation and also gives more accurate VaRs when this assumption holds.
- The method based on sphericity may yield inconsistent VaR estimators when this assumption is in failure.
- The FHS method performs well in both cases and outperforms the first approach in the absence of sphericity.

Conclusions: multivariate approaches

- For both approaches, asymptotic CIs for the conditional VaR can be built.
  ⇒ allows to visualize on the same graph both market and estimation risks.
- Exploiting the sphericity simplifies estimation and also gives more accurate VaRs when this assumption holds.
- The method based on sphericity may yield inconsistent VaR estimators when this assumption is in failure.
- The FHS method performs well in both cases and outperforms the first approach in the absence of sphericity.

Thanks for your attention!

References

Bauwens, L., Hafner, C.M. and S. Laurent (2012)
Handbook of Volatility Models and Their Applications. Wiley.

Bauwens, L., Laurent, S. and J.V.K. Rombouts (2006)
Multivariate GARCH models: a survey. *Journal of Applied Econometrics* 21, 79–109.

Escanciano, J.C., and J. Olmo (2010)
Backtesting parametric VaR with estimation risk. *Journal of Business and Economic Statistics* 28, 36–51.

Kuester, K., Mittnik, S. and M. Paolella (2006)
Value-at-Risk prediction: a comparison of alternative strategies. *Journal of Financial Econometrics* 4, 53–89.

Rombouts, J.V.K. and M. Verbeek (2009)
Evaluating portfolio Value-at-Risk using semi-parametric GARCH models. *Quantitative Finance* 17, 737–745.

Vector GARCH model

$$\begin{aligned} \boldsymbol{\epsilon}_t &= \boldsymbol{H}_t^{1/2}\boldsymbol{\eta}_t, \quad \boldsymbol{H}_t \text{ positive definite, } (\boldsymbol{\eta}_t) \text{ iid } (\mathbf{0},\boldsymbol{I}) \\ \text{vech}(\boldsymbol{H}_t) &= \boldsymbol{\omega} + \sum_{i=1}^{q} \boldsymbol{A}^{(i)}\text{vech}(\boldsymbol{\epsilon}_{t-i}\boldsymbol{\epsilon}'_{t-i}) + \sum_{j=1}^{p} \boldsymbol{B}^{(j)}\text{vech}(\boldsymbol{H}_{t-j}) \end{aligned}$$

- The most direct generalization of univariate GARCH
- Positivity conditions are difficult to obtain
- No explicit stationarity conditions

BEKK-GARCH model

Engle and Kroner (1995), Comte and Lieberman (2003)

$$\begin{cases} \boldsymbol{\epsilon}_t = \boldsymbol{H}_t^{1/2}\boldsymbol{\eta}_t, & (\boldsymbol{\eta}_t) \text{ iid } (\boldsymbol{0},\boldsymbol{I}) \\ \boldsymbol{H}_t = \boldsymbol{\Omega} + \sum_{i=1}^{q}\sum_{k=1}^{K} \boldsymbol{A}_{ik}\boldsymbol{\epsilon}_{t-i}\boldsymbol{\epsilon}'_{t-i}\boldsymbol{A}'_{ik} + \sum_{j=1}^{p}\sum_{k=1}^{K} \boldsymbol{B}_{jk}\boldsymbol{H}_{t-j}\boldsymbol{B}'_{jk} \end{cases}$$

- Coefficients of a BEKK representation are difficult to interpret
- Positivity conditions are simple. Identifiability of a BEKK representation requires additional constraints.
- Stationarity conditions exist (Boussama, Fuchs, Stelzer, 2011) but no explicit solution can be exhibited

Constant Conditional Correlation (CCC) model

Bollerslev (1990); Extended CCC by Jeantheau (1998)

$$\underline{\boldsymbol{h}}_t = \begin{pmatrix} h_{11,t} \\ \vdots \\ h_{mm,t} \end{pmatrix}, \quad \boldsymbol{D}_t = \text{diag}\left(h_{11,t}^{1/2}, \dots, h_{mm,t}^{1/2}\right), \quad \underline{\boldsymbol{\epsilon}}_t = \begin{pmatrix} \epsilon_{1t}^2 \\ \vdots \\ \epsilon_{mt}^2 \end{pmatrix}.$$

$$\begin{cases} \boldsymbol{\epsilon}_t = \boldsymbol{H}_t^{1/2} \boldsymbol{\eta}_t, \qquad \boldsymbol{H}_t = \boldsymbol{D}_t \boldsymbol{R} \boldsymbol{D}_t, \quad \boldsymbol{R}\text{: correlation matrix} \\ \underline{\boldsymbol{h}}_t = \boldsymbol{\omega} + \sum_{i=1}^{q} \mathbf{A}_i \underline{\boldsymbol{\epsilon}}_{t-i} + \sum_{j=1}^{p} \mathbf{B}_j \underline{\boldsymbol{h}}_{t-j} \end{cases}$$

- Simple conditions ensuring the positive definiteness of H_t.
- Explicit stationarity condition (of the form $\gamma < 0$...)
- The assumption of CCC can be too restrictive

Dynamic Conditional Correlation (DCC) model

Engle (2002)

$$\boldsymbol{H}_t = \boldsymbol{D}_t \boldsymbol{R}_t \boldsymbol{D}_t, \qquad \boldsymbol{R}_t = (\operatorname{diag} \boldsymbol{Q}_t)^{-1/2} \boldsymbol{Q}_t (\operatorname{diag} \boldsymbol{Q}_t)^{-1/2},$$

where $\boldsymbol{\eta}_t^* = \boldsymbol{D}_t^{-1} \boldsymbol{\epsilon}_t$ and

$$\boldsymbol{Q}_t = (1 - \alpha - \beta) \boldsymbol{S} + \alpha \boldsymbol{\eta}_{t-1}^* \boldsymbol{\eta}_{t-1}^{*'} + \beta \boldsymbol{Q}_{t-1},$$

where $\alpha, \beta \geq 0, \alpha + \beta < 1$, $\boldsymbol{S}$ is a correlation matrix

- The existence of strictly stationary solution is a complex issue (recent PhD thesis by Malongo, 2014)
- No asymptotic theory of estimation exists
- Incorrect interpretation of $\boldsymbol{S}$ as $\mathrm{Var}(\boldsymbol{\eta}_t^*)$ and $\boldsymbol{Q}_t$ as $\mathrm{Var}_{t-1}(\boldsymbol{\eta}_t^*)$.

Dynamic Conditional Correlation (DCC) model

Corrected DCC (Aielli (2013)

$$\boldsymbol{Q}_t = (1-\alpha-\beta)\boldsymbol{S} + \alpha \boldsymbol{Q}_{t-1}^{*1/2}\boldsymbol{\eta}_{t-1}^{*}\boldsymbol{\eta}_{t-1}^{*'}\boldsymbol{Q}_{t-1}^{*1/2} + \beta \boldsymbol{Q}_{t-1},$$

where $\boldsymbol{Q}_t^* = \mathrm{diag}(\boldsymbol{Q}_t)$.

- Identifiability constraint: diag$(\boldsymbol{S}) = \boldsymbol{I}_m$.
- Parcimony but the $m(m-1)/2$ conditional correlations have the same dynamic structure.

Return

Example: Linear SRE on the derivatives of H_t

BEKK-GARCH(1,1) model:

$$\epsilon_t = H_t^{1/2}\eta_t, \qquad H_t = C_0 + A_0\epsilon_{t-1}\epsilon_{t-1}'A_0' + B_0 H_{t-1} B_0'$$

Let $\theta = (\text{vec}(A)', \text{vec}(B)', \text{vec}(C)')'$. For $j = 1, \dots, 3d$,

$$\begin{aligned}\frac{\partial \text{vec}(H_t)}{\partial \theta_j} &= \frac{\partial \text{vec}(C)}{\partial \theta_j} + \frac{\partial (A \otimes A)}{\partial \theta_j}\text{vec}(\epsilon_t \epsilon_t') \\ &+ \frac{\partial (B \otimes B)}{\partial \theta_j}\text{vec}(H_{t-1}) + (B \otimes B)\frac{\partial \text{vec}(H_{t-1})}{\partial \theta_j},\end{aligned}$$

allows to compute recursively the derivatives of H_t (for some initial values).

We note that $\Sigma_t \frac{\partial \Sigma_t}{\partial \theta_i} + \frac{\partial \Sigma_t}{\partial \theta_i}\Sigma_t = \frac{\partial H_t}{\partial \theta_i}$. Thus

$$(I_m \otimes \Sigma_t + \Sigma_t \otimes I_m)\,\text{vec}\left(\frac{\partial \Sigma_t}{\partial \theta_i}\right) = \text{vec}\left(\frac{\partial H_t}{\partial \theta_i}\right).$$

Return

Example

An equally weighted portfolio of 3 assets:

$$V_t = \sum_{i=1}^{3} p_{it}.$$

The vector of the log-returns

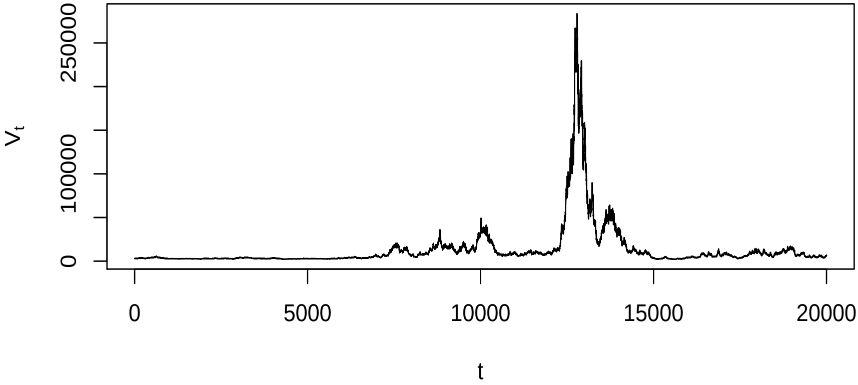
$$\boldsymbol{y}_t \sim \text{iid } \mathcal{N}(\boldsymbol{0}, \boldsymbol{DRD}),$$

with

$$\boldsymbol{D} = \begin{pmatrix} 0.01 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & 0.04 \end{pmatrix}, \quad \boldsymbol{R} = \begin{pmatrix} 1 & -0.855 & 0.855 \\ -0.855 & 1 & -0.810 \\ 0.855 & -0.810 & 1 \end{pmatrix}.$$

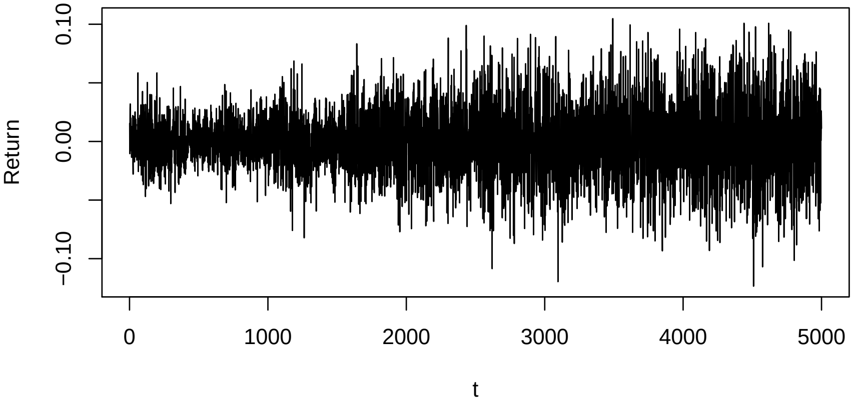
The composition of the log-return portfolio is not constant: $a_{i,t-1} = \frac{p_{i,t-1}}{\sum_{j=1}^{3} p_{j,t-1}}$.

A trajectory of (V_t)



The process (V_t) is non stationary.

A trajectory of (r_t)



The return process (r_t) (also non stationary)

Time-varying composition of the portfolio

Time-varying composition of the portfolio

Return

DCC-GARCH model for the individual returns

$$\begin{cases} \boldsymbol{\epsilon}_t = \boldsymbol{\Sigma}_t \boldsymbol{\eta}_t, \qquad \boldsymbol{\Sigma}_t^2 = \boldsymbol{D}_t \boldsymbol{R}_t \boldsymbol{D}_t, \quad \boldsymbol{D}_t^2 = \text{diag}(\underline{\boldsymbol{h}}_t), \\ \underline{\boldsymbol{h}}_t = \boldsymbol{\omega}_0 + \mathbf{A}_0 \underline{\boldsymbol{\epsilon}}_{t-1} + \mathbf{B}_0, \underline{\boldsymbol{h}}_{t-1}, \quad \underline{\boldsymbol{\epsilon}}_t = \begin{pmatrix} \epsilon_{1t}^2 \\ \vdots \\ \epsilon_{mt}^2 \end{pmatrix} \end{cases}$$

where $\mathbf{B}_0$ is diagonal, and the correlation $\boldsymbol{R}_t$ follows the cDCC model (Engle (2002), Aielli (2013))

$$\boldsymbol{R}_t = \boldsymbol{Q}_t^{*-1/2} \boldsymbol{Q}_t \boldsymbol{Q}_t^{*-1/2},$$

$$\boldsymbol{Q}_t = (1-\alpha_0-\beta_0)\boldsymbol{S}_0 + \alpha_0 \boldsymbol{Q}_{t-1}^{*1/2} \boldsymbol{\eta}_{t-1}^{*} \boldsymbol{\eta}_{t-1}^{*'} \boldsymbol{Q}_{t-1}^{*1/2} + \beta_0 \boldsymbol{Q}_{t-1},$$

where $\alpha_0, \beta_0 \geq 0, \alpha_0 + \beta_0 < 1$, $\boldsymbol{S}_0$ is a correlation matrix, $\boldsymbol{Q}_t^*$ is the diagonal matrix with the same diagonal elements as $\boldsymbol{Q}_t$, and $\boldsymbol{\eta}_t^* = \boldsymbol{D}_t^{-1} \boldsymbol{\epsilon}_t$.

Designs of the numerical experiments

Table: Design of Monte Carlo experiments.

	ω_0'	$(\text{vec}\boldsymbol{A}_0)'$	diag$\boldsymbol{B}_0$	$\boldsymbol{S}_0(1,2)$	α	β	$P_{\boldsymbol{\eta}}$
A	$(10^{-6}, 4\times10^{-6})$	$(0.01, 0.01, 0.01, 0.07)$	$(0, 0.92)$	0.7	0.04	0.95	$\mathscr{N}(0, \boldsymbol{I}_2)$
B	$(10^{-6}, 4\times10^{-6})$	$(0.01, 0.01, 0.01, 0.07)$	$(0, 0.92)$	0.7	0.04	0.95	$\mathscr{S}t_7$
C	$(10^{-6}, 4\times10^{-6})$	$(0.01, 0.01, 0.01, 0.07)$	$(0, 0.92)$	0	0	0	$\mathscr{N}(0, \boldsymbol{I}_2)$
D	$(10^{-6}, 4\times10^{-6})$	$(0.01, 0.01, 0.01, 0.07)$	$(0, 0.92)$	0	0	0	$\mathscr{S}t_7$
E	$(10^{-5}, 10^{-5})$	$(0.07, 0.00, 0.00, 0.07)$	$(0.92, 0.92)$	0.7	0.04	0.95	$\mathscr{N}(0, \boldsymbol{I}_2)$
F	$(10^{-5}, 10^{-5})$	$(0.07, 0.00, 0.00, 0.07)$	$(0.92, 0.92)$	0.7	0.04	0.95	$\mathscr{S}t_7$
G	$(10^{-5}, 10^{-5})$	$(0.07, 0.00, 0.00, 0.07)$	$(0.92, 0.92)$	0	0	0	$\mathscr{N}(0, \boldsymbol{I}_2)$
H	$(10^{-5}, 10^{-5})$	$(0.07, 0.00, 0.00, 0.07)$	$(0.92, 0.92)$	0	0	0	$\mathscr{S}t_7$

Designs A*-H* are the same as Designs A-H, except that $P_{\boldsymbol{\eta}}$ follows an asymmetric AEPD (introduced by Zhu and Zinde-Walsh (2009)).

◂ Numerical experiments